satisfies (1.73) in the steady state.

As a final observation we note that the boson commutation relation is preserved in time – at least in the mean, which is all we can say in the Schrödinger picture. Using the initial time commutator we find

$$\langle [a, a^\dagger](t)\rangle = \mathrm{tr}\{[a, a^\dagger]\rho(t)\} = \mathrm{tr}\{\rho(t)\} = 1;$$

it is readily shown that (1.73) preserves the trace of the density operator.

1.5 Two-Time Averages and the Quantum Regression Theorem

We have developed a formalism which allows us, in principle, to solve for the density operator (reduced density operator) for a system interacting with a reservoir. From this density operator we can obtain time-dependent expectation values for any operator acting in the Hilbert space of the system S. What, however, about products of operators evaluated at two different times? Of particular interest, for example, will be the first-order and second-order correlation functions of the electromagnetic field. For a single mode these are given by

$$G^{(1)}(t, t+\tau) \propto \langle a^\dagger(t) a(t+\tau)\rangle,$$
$$G^{(2)}(t, t+\tau) \propto \langle a^\dagger(t) a^\dagger(t+\tau) a(t+\tau) a(t)\rangle.$$

The first-order correlation function is required for calculating the spectrum of the field. The second-order correlation function gives information about the photon statistics and describes photon bunching and antibunching.

Note 1.4 It may seem a strange talking about the spectrum of a single mode field since we normally associate a single mode with a single frequency. Here we are dealing, however, with what should more correctly be called a quasimode – a mode defined in a *lossy* optical cavity, which therefore has a finite linewidth.

Clearly, averages involving two times cannot be calculated directly from the master equation – at least, not without a little extra thought. We need to return to the microscopic picture of system plus reservoir. At this level two-time averages are defined in the usual way in the Heisenberg representation. Our objective, then, is to derive a relationship that allows us to calculate these averages at the macroscopic level using the master equation for the reduced density operator alone; thus, in some approximate way we wish to carry out the trace over reservoir variables explicitly, as we did in deriving the master equation itself. The result we obtain is known as the quantum regression theorem and is attributed to Lax [1.20, 1.21]. We will not follow Lax in detail, but our method is fundamentally the same as his.

1.5.1 Formal Results

Recall our microscopic formulation of system S coupled to reservoir R. The Hamiltonian for the composite system $S \oplus R$ takes the form given in (1.16). The density operator is designated $\chi(t)$ and satisfies Schrödinger's equation (1.19). Our derivation of the master equation has given us an equation for the reduced density operator (1.17), which we will now write formally as

$$\dot{\rho} = \mathcal{L}\rho; \tag{1.81}$$

$\mathcal{L}$ is a generalized Liouvillian, a "superoperator" in the language of the Brussells-Austin group [1.22]; $\mathcal{L}$ operates on operators rather than on states. For the damped harmonic oscillator, from (1.73), the action of $\mathcal{L}$ on an arbitrary operator $\hat{O}$ is defined by the equation

$$\begin{aligned}\mathcal{L}\hat{O} \equiv & -i\omega_0[a^\dagger a, \hat{O}] + \frac{\gamma}{2}(2a\hat{O}a^\dagger - a^\dagger a\hat{O} - \hat{O}a^\dagger a) \\ & + \gamma\bar{n}(a\hat{O}a^\dagger + a^\dagger\hat{O}a - a^\dagger a\hat{O} - \hat{O}aa^\dagger).\end{aligned} \tag{1.82}$$

Within the microscopic formalism multi-time averages are straightforwardly defined in the Heisenberg picture. In particular, the average of a product of operators evaluated at two different times is given by

$$\langle \hat{O}_1(t)\hat{O}_2(t')\rangle = \mathrm{tr}_{S\oplus R}[\chi(0)\hat{O}_1(t)\hat{O}_2(t')], \tag{1.83}$$

where $\hat{O}_1$ and $\hat{O}_2$ are any two system operators. These operators satisfy the Heisenberg equations of motion

$$\dot{\hat{O}}_1 = \frac{1}{i\hbar}[\hat{O}_1, H], \tag{1.84a}$$

$$\dot{\hat{O}}_2 = \frac{1}{i\hbar}[\hat{O}_2, H], \tag{1.84b}$$

with the formal solutions

$$\hat{O}_1(t) = e^{(i/\hbar)Ht}\hat{O}_1(0)e^{-(i/\hbar)Ht}, \tag{1.85a}$$

$$\hat{O}_2(t') = e^{(i/\hbar)Ht'}\hat{O}_2(0)e^{-(i/\hbar)Ht'}. \tag{1.85b}$$

From (1.19), the formal solution for χ gives

$$\chi(0) = e^{(i/\hbar)Ht}\chi(t)e^{-(i/\hbar)Ht}. \tag{1.86}$$

We substitute these formal solutions into (1.83) and use the cyclic property of the trace to obtain

$$\begin{aligned}\langle \hat{O}_1(t)\hat{O}_2(t')\rangle &= \mathrm{tr}_{S\oplus R}\Big[e^{(i/\hbar)Ht}\chi(t)\hat{O}_1(0)e^{(i/\hbar)H(t'-t)}\hat{O}_2(0)e^{-(i/\hbar)Ht'}\Big] \\ &= \mathrm{tr}_{S\oplus R}\Big[\hat{O}_2(0)e^{-(i/\hbar)H(t'-t)}\chi(t)\hat{O}_1(0)e^{(i/\hbar)H(t'-t)}\Big] \\ &= \mathrm{tr}_S\Big\{\hat{O}_2(0)\mathrm{tr}_R\Big[e^{-(i/\hbar)H(t'-t)}\chi(t)\hat{O}_1(0)e^{(i/\hbar)H(t'-t)}\Big]\Big\}.\end{aligned} \tag{1.87}$$

In the final step we have used the fact that $\hat{O}_2$ is an operator in the Hilbert space of S alone.

We now specialize to the case $t' \geq t$ and define

$$\tau \equiv t' - t, \tag{1.88}$$

$$\chi_{\hat{O}_1}(\tau) \equiv e^{-(i/\hbar)H\tau}\chi(t)\hat{O}_1(0)e^{(i/\hbar)H\tau}. \tag{1.89}$$

Clearly, $\chi_{\hat{O}_1}$ satisfies the equation

$$\frac{d\chi_{\hat{O}_1}}{d\tau} = \frac{1}{i\hbar}\Big[H, \chi_{\hat{O}_1}\Big] \tag{1.90}$$

with

$$\chi_{\hat{O}_1}(0) = \chi(t)\hat{O}_1(0). \tag{1.91}$$

If we are to eliminate explicit reference to the reservoir in (1.87), we need to evaluate the reservoir trace over $\chi_{\hat{O}_1}(\tau)$ to obtain the reduced operator

$$\rho_{\hat{O}_1}(\tau) \equiv \mathrm{tr}_R\Big[\chi_{\hat{O}_1}(\tau)\Big], \tag{1.92}$$

where

$$\rho_{\hat{O}_1}(0) = \mathrm{tr}_R[\chi(t)\hat{O}_1(0)] = \mathrm{tr}_R[\chi(t)]\hat{O}_1(0) = \rho(t)\hat{O}_1(0); \tag{1.93}$$

notice that $\rho_{\hat{O}_1}(\tau)$ is just the term $\mathrm{tr}_R[\cdots]$ inside the curly brackets in (1.87). If we then assume that $\chi(t)$ factorizes as $\rho(t)R_0$, in the spirit of (1.29), from (1.91) and (1.93) we can write

$$\chi_{\hat{O}_1}(0) = R_0[\rho(t)\hat{O}_1(0)] = R_0\,\rho_{\hat{O}_1}(0). \tag{1.94}$$

Equations (1.90), (1.92), and (1.94) are now equivalent to (1.19), (1.17), and (1.25) – namely, to the starting equations in our derivation of the master equation. We can find an equation for $\rho_{\hat{O}_1}(\tau)$ in the Born-Markov approximation following a completely analogous course to that followed in Sects. 1.3 and 1.4. Since (1.19) and (1.90) contain the same Hamiltonian H, using the formal notation of (1.81), we arrive at the equation

$$\frac{d\rho_{\hat{O}_1}}{d\tau} = \mathcal{L}\rho_{\hat{O}_1}, \tag{1.95}$$

with solution

$$\rho_{\hat{O}_1}(\tau) = e^{\mathcal{L}\tau}\left[\rho_{\hat{O}_1}(0)\right] = e^{\mathcal{L}\tau}[\rho(t)\hat{O}_1(0)]. \tag{1.96}$$

When we substitute for $\rho_{\hat{O}_1}(\tau)$ in (1.87), we have ($\tau \geq 0$)

$$\langle \hat{O}_1(t)\hat{O}_2(t+\tau)\rangle = \mathrm{tr}_S\{\hat{O}_2(0)e^{\mathcal{L}\tau}[\rho(t)\hat{O}_1(0)]\}. \tag{1.97}$$

Exercise 1.3 Follow the same procedure to obtain ($\tau \geq 0$)

$$\langle \hat{O}_1(t+\tau)\hat{O}_2(t)\rangle = \mathrm{tr}_S\{\hat{O}_1(0)e^{\mathcal{L}\tau}[\hat{O}_2(0)\rho(t)]\}. \tag{1.98}$$

Equations (1.97) and (1.98) give formal statements of the *quantum regression theorem* for two-time averages. To calculate a correlation function $\langle \hat{O}_1(t)\hat{O}_2(t')\hat{O}_3(t)\rangle$ we cannot use (1.97) and (1.98) because noncommuting operators do not allow the reordering necessary to bring $\hat{O}_1(t)$ next to $\hat{O}_3(t)$. We may, however, generalize the approach taken above. Specifically, we have

$$\begin{aligned}
&\langle \hat{O}_1(t)\hat{O}_2(t')\hat{O}_3(t)\rangle \\
&\quad = \mathrm{tr}_{S\oplus R}\Big[e^{(i/\hbar)Ht}\chi(t)\hat{O}_1(0)e^{(i/\hbar)H(t'-t)}\hat{O}_2(0)e^{-(i/\hbar)H(t'-t)} \\
&\qquad \times\hat{O}_3(0)e^{-(i/\hbar)Ht}\Big] \\
&\quad = \mathrm{tr}_{S\oplus R}\Big[\hat{O}_2(0)e^{-(i/\hbar)H(t'-t)}\hat{O}_3(0)\chi(t)\hat{O}_1(0)e^{(i/\hbar)H(t'-t)}\Big] \\
&\quad = \mathrm{tr}_S\Big\{\hat{O}_2(0)\mathrm{tr}_R\Big[e^{-(i/\hbar)H(t'-t)}\hat{O}_3(0)\chi(t)\hat{O}_1(0)e^{(i/\hbar)H(t'-t)}\Big]\Big\}.
\end{aligned} \tag{1.99}$$

Defining

$$\chi_{\hat{O}_3\hat{O}_1}(\tau) \equiv e^{-(i/\hbar)H\tau}\hat{O}_3(0)\chi(t)\hat{O}_1(0)e^{(i/\hbar)H\tau} \tag{1.100}$$

and

$$\rho_{\hat{O}_3\hat{O}_1}(\tau) \equiv \mathrm{tr}_R\left[\chi_{\hat{O}_3\hat{O}_1}(\tau)\right] \tag{1.101}$$

as analogs of (1.89) and (1.92), we can proceed as before to the result ($\tau \geq 0$)

$$\langle \hat{O}_1(t)\hat{O}_2(t+\tau)\hat{O}_3(t)\rangle = \mathrm{tr}_S\{\hat{O}_2(0)e^{\mathcal{L}\tau}[\hat{O}_3(0)\rho(t)\hat{O}_1(0)]\}. \tag{1.102}$$

Equations (1.97) and (1.98) are, in fact, just special cases of (1.102) with either $\hat{O}_1(t)$ or $\hat{O}_3(t)$ set equal to the unit operator.

1.5.2 Quantum Regression Theorem for a Complete Set of Operators

It is possible to work directly with the rather formal expressions derived above. The formal expressions can also be reduced, however, to a more familiar form [1.20], which is often more convenient for doing calculations. Essentially, we will find that the equations of motion for expectation values of system operators (one-time averages) are also the equations of motion for correlation functions (two-time averages).

We begin by assuming that there exists a complete set of system operators $\hat{A}_\mu$, $\mu = 1, 2, \ldots$, in the following sense: that for an arbitrary operator $\hat{O}$, and for each $\hat{A}_\mu$,

$$\mathrm{tr}_S[\hat{A}_\mu(\mathcal{L}\hat{O})] = \sum_\lambda M_{\mu\lambda}\mathrm{tr}_S(\hat{A}_\lambda\hat{O}), \tag{1.103}$$

where the $M_{\mu\lambda}$ are constants. In particular, from this it follows that

$$\begin{aligned}\langle\dot{\hat{A}}_\mu\rangle = \mathrm{tr}_S(\hat{A}_\mu\dot{\rho}) &= \mathrm{tr}_S[\hat{A}_\mu(\mathcal{L}\rho)] \\ &= \sum_\lambda M_{\mu\lambda}\mathrm{tr}_S(\hat{A}_\lambda\rho) \\ &= \sum_\lambda M_{\mu\lambda}\langle\hat{A}_\lambda\rangle. \end{aligned}\tag{1.104}$$

Thus, expectation values $\langle\hat{A}_\mu\rangle$, $\mu = 1, 2, \ldots$, obey a coupled set of linear equations with the evolution matrix $\boldsymbol{M}$ defined by the $M_{\mu\lambda}$ that appear in (1.103). In vector notation,

$$\langle\dot{\hat{\boldsymbol{A}}}\rangle = \boldsymbol{M}\langle\hat{\boldsymbol{A}}\rangle, \tag{1.105}$$

where $\hat{\boldsymbol{A}}$ is the column vector of operators $\hat{A}_\mu$, $\mu = 1, 2, \ldots$. Now, using (1.97) and (1.103) ($\tau \geq 0$):

$$\begin{aligned}\frac{d}{d\tau}\langle\hat{O}_1(t)\hat{A}_\mu(t+\tau)\rangle &= \mathrm{tr}_S\{\hat{A}_\mu(0)(\mathcal{L}e^{\mathcal{L}\tau}[\rho(t)\hat{O}_1(0)])\} \\ &= \sum_\lambda M_{\mu\lambda}\mathrm{tr}_S\{\hat{A}_\lambda(0)e^{\mathcal{L}\tau}[\rho(t)\hat{O}_1(0)]\} \\ &= \sum_\lambda M_{\mu\lambda}\langle\hat{O}_1(t)\hat{A}_\lambda(t+\tau)\rangle, \end{aligned}\tag{1.106}$$

or,

$$\frac{d}{d\tau}\langle\hat{O}_1(t)\hat{\boldsymbol{A}}(t+\tau)\rangle = \boldsymbol{M}\langle\hat{O}_1(t)\hat{\boldsymbol{A}}(t+\tau)\rangle, \tag{1.107}$$

where $\hat{O}_1$ can be any system operator, not necessarily one of the $\hat{A}_\mu$. This result is just what would be obtained by removing the angular brackets from (1.105) (written with $t \to t+\tau$, and $\cdot \equiv d/dt \to d/d\tau$), multiplying on the left by $\hat{O}_1(t)$, and then replacing the angular brackets. Hence, for each operator $\hat{O}_1$, the set of correlation functions $\langle\hat{O}_1(t)\hat{A}_\mu(t+\tau)\rangle$, $\mu=1,2,\ldots$, with $\tau\geq 0$, satisfies the same equations (as functions of τ) as do the averages $\langle\hat{A}_\mu(t+\tau)\rangle$. This is perhaps the more familiar statement of the *quantum regression theorem*.

Exercise 1.4 For $\tau\geq 0$ show that

$$\frac{d}{d\tau}\langle\hat{\boldsymbol{A}}(t+\tau)\hat{O}_2(t)\rangle = \boldsymbol{M}\langle\hat{\boldsymbol{A}}(t+\tau)\hat{O}_2(t)\rangle. \tag{1.108}$$

Thus, we can also multiply (1.105) on the right by $\hat{O}_2(t)$, inside the average. Also show that

$$\frac{d}{d\tau}\langle\hat{O}_1(t)\hat{\boldsymbol{A}}(t+\tau)\hat{O}_2(t)\rangle = \boldsymbol{M}\langle\hat{O}_1(t)\hat{\boldsymbol{A}}(t+\tau)\hat{O}_2(t)\rangle. \tag{1.109}$$

It may appear that this form of the quantum regression theorem is quite restricted, since its derivation relies on the existence of a set of operators $\hat{A}_\mu$, $\mu=1,2,\ldots$, for which (1.103) holds. We can show that this is always so, however, if a discrete basis $|n\rangle$, $n=1,2,\ldots$, exists; although, in general, the complete set of operators may be very large. Consider the operators

$$\hat{A}_\mu = \hat{A}_{nm} \equiv |n\rangle\langle m|. \tag{1.110}$$

Then

$$\begin{aligned}
\mathrm{tr}_S[\hat{A}_{nm}(\mathcal{L}\hat{O})] &= \mathrm{tr}_S\big[|n\rangle\langle m|(\mathcal{L}\hat{O})\big] \\
&= \langle m|(\mathcal{L}\hat{O})|n\rangle \\
&= \langle m|\left(\mathcal{L}\sum_{n',m'}|n'\rangle\langle m'|\langle n'|\hat{O}|m'\rangle\right)|n\rangle \\
&= \sum_{n',m'}\langle m|\big(\mathcal{L}|n'\rangle\langle m'|\big)|n\rangle\langle n'|\hat{O}|m'\rangle \\
&= \sum_{n',m'}\langle m|\big(\mathcal{L}|n'\rangle\langle m'|\big)|n\rangle\,\mathrm{tr}_S\big(|m'\rangle\langle n'|\hat{O}\big) \\
&= \sum_{n',m'} M_{nm;n'm'}\,\mathrm{tr}_S(\hat{A}_{n'm'}\hat{O}),
\end{aligned} \tag{1.111}$$

with

$$M_{nm;n'm'} \equiv \langle m|\Big(\mathcal{L}|m'\rangle\langle n'|\Big)|n\rangle. \tag{1.112}$$

In the last step we have interchanged the indices n' and m'. Equation (1.111) gives an expansion in the form of (1.103). The complete set of operators includes all the outer products $|n\rangle\langle m|$, $n=1,2,\ldots$, $m=1,2,\ldots$; this may be a small number of operators, a large but finite number of operators, or a double infinity of operators in the case of the Fock state basis.

1.5.3 Correlation Functions for the Damped Harmonic Oscillator

We will conclude our discussion of two-time averages with two simple examples based on the equations for expectation values for the damped harmonic oscillator [Eqs. (1.78) and (1.79)]. We first calculate the first-order correlation function $\langle a^\dagger(t)a(t+\tau)\rangle$. Equation (1.78) gives the equation of motion for the mean oscillator amplitude:

$$\langle\dot{a}\rangle = -\left(\frac{\gamma}{2} + i\omega_0\right)\langle a\rangle. \tag{1.113}$$

Then, with $\hat{A}_1 = a$ and $\hat{O}_1 = a^\dagger$, from (1.105) and (1.107), we may write

$$\frac{d}{d\tau}\langle a^\dagger(t)a(t+\tau)\rangle = -\left(\frac{\gamma}{2} + i\omega_0\right)\langle a^\dagger(t)a(t+\tau)\rangle. \tag{1.114}$$

Thus,

$$\begin{aligned}\langle a^\dagger(t)a(t+\tau)\rangle &= \langle a^\dagger(t)a(t)\rangle e^{-(\gamma/2+i\omega_0)\tau}\\ &= \left[\langle\hat{n}(0)\rangle e^{-\gamma t} + \bar{n}(1-e^{-\gamma t})\right]e^{-(\gamma/2+i\omega_0)\tau},\end{aligned} \tag{1.115}$$

where the last line follows from (1.80). If the oscillator describes a lossy cavity mode, in the long-time limit the Fourier transform of the first-order correlation function

$$\langle a^\dagger(0)a(\tau)\rangle_{\rm ss} \equiv \lim_{t\to\infty}\langle a^\dagger(t)a(t+\tau)\rangle = \bar{n}e^{-(\gamma/2+i\omega_0)\tau} \tag{1.116}$$

gives the spectrum of the light at the cavity output. This is clearly a Lorentzian with width γ (full-width at half-maximum).

Note 1.5 This statement about the spectrum of the light at the cavity output is not strictly correct for the lossy cavity model as we have described it. The reason is that we have taken the environment outside the cavity to be in thermal equilibrium at temperature T (it is the environment that is modeled by the reservoir). Given this, the light detected in the cavity output will be a sum of transmitted light – light that passes from inside the cavity, through the cavity output mirror, into the environment – and thermal radiation reflected from the outside of the output mirror. Calculating the spectrum at the cavity output for this situation is more involved (Sect. 7.3.4). Physically, however, the result is clear; the spectrum must be a blackbody spectrum. The Lorentzian spectrum obtained from (1.116) would be observed, as filtered thermal radiation, for a cavity coupled to two reservoirs, one at temperature T and the other at zero temperature. If the bandwidth for coupling to the reservoir at temperature T is much larger than for coupling to the zero temperature reservoir, the master equation (1.73) is basically unchanged. Light emitted into the zero temperature reservoir then shows the Lorentzian spectrum obtained from the Fourier transform of (1.116).

For a second example we calculate the second-order correlation function $\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\rangle = \langle a^\dagger(t)\hat{n}(t+\tau)a(t)\rangle$. Writing (1.79) in the form

$$\frac{d}{dt}\begin{pmatrix}\langle\hat{n}\rangle\\ \bar{n}\end{pmatrix} = \begin{pmatrix}-\gamma & \gamma\\ 0 & 0\end{pmatrix}\begin{pmatrix}\langle\hat{n}\rangle\\ \bar{n}\end{pmatrix}, \tag{1.117}$$

we set $\hat{A}_1 = \hat{n} = a^\dagger a$ and $\hat{A}_2 = \bar{n}$ (a constant). Then, from (1.105) and (1.109), with $\hat{O}_1 = a^\dagger$ and $\hat{O}_2 = a$,

$$\frac{d}{d\tau}\begin{pmatrix}\langle a^\dagger(t)\hat{n}(t+\tau)a(t)\rangle\\ \bar{n}\langle\hat{n}(t)\rangle\end{pmatrix} = \begin{pmatrix}-\gamma & \gamma\\ 0 & 0\end{pmatrix}\begin{pmatrix}\langle a^\dagger(t)\hat{n}(t+\tau)a(t)\rangle\\ \bar{n}\langle\hat{n}(t)\rangle\end{pmatrix}. \tag{1.118}$$

Thus,

$$\langle a^\dagger(t)\hat{n}(t+\tau)a(t)\rangle = \langle a^\dagger(t)\hat{n}(t)a(t)\rangle e^{-\gamma\tau} + \bar{n}\langle\hat{n}(t)\rangle(1-e^{-\gamma\tau}). \tag{1.119}$$

We obtained an expression for $\langle\hat{n}(t)\rangle$ in (1.80). The calculation of $\langle a^\dagger(t)\hat{n}(t)\, a(t)\rangle$ is left as an exercise:

Exercise 1.5 Derive an equation of motion for the expectation value $\langle a^\dagger(t)\, \hat{n}(t)a(t)\rangle = \langle a^{\dagger 2}(t)a^2(t)\rangle$ from the master equation (1.73) and show that

$$\begin{aligned}\langle a^\dagger(t)\hat{n}(t)a(t)\rangle = {}& \left[\langle\hat{n}^2(0)\rangle - \langle\hat{n}(0)\rangle\right]e^{-2\gamma t} + 2\bar{n}(1-e^{-\gamma t})\\ &\times\left[2\langle\hat{n}(0)\rangle e^{-\gamma t} + \bar{n}(1-e^{-\gamma t})\right].\end{aligned} \tag{1.120}$$

Now, substituting from (1.80) and (1.120) into (1.119),

$$\begin{aligned}&\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\rangle\\ &\quad = \big\{\left[\langle\hat{n}^2(0)\rangle - \langle\hat{n}(0)\rangle\right]e^{-2\gamma t} + 2\bar{n}(1-e^{-\gamma t})\big[2\langle\hat{n}(0)\rangle e^{-\gamma t}\\ &\qquad +\bar{n}(1-e^{-\gamma t})\big]\big\}e^{-\gamma\tau} + \bar{n}\left[\langle\hat{n}(0)\rangle e^{-\gamma t} + \bar{n}(1-e^{-\gamma t})\right](1-e^{-\gamma\tau}).\end{aligned} \tag{1.121}$$

In the long-time limit, the second-order correlation function is

$$\begin{aligned}\langle a^\dagger(0)a^\dagger(\tau)a(\tau)a(0)\rangle_{ss} &\equiv \lim_{t\to\infty}\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\rangle\\ &= \bar{n}^2(1+e^{-\gamma\tau}).\end{aligned} \tag{1.122}$$

This expression describes the well-known Hanbury-Brown-Twiss effect, or photon bunching, for thermal light [1.23]; at zero delay the correlation function has twice the value it has for long delays ($\gamma\tau \gg 1$).

Note 1.6 The correlation time, $1/\gamma$, in (1.122) holds for filtered thermal light in accord with the comments in Note 1.5.